Journal of Culture and Values in Education

https://cultureandvalues.org

E-ISSN: 2590-342X

Volume: 7 Issue: 2 2024

pp. 70-100

Exploring Students' Misconceptions in Learning Mathematical Proof Techniques at Debark University in Ethiopia

Aschale Moges Belay*[a], France Machaba[b], & Tšhegofatšo Phuti Makgakga[b]

* Corresponding author:
Email:17217059@mylife.unisa.ac.za

a. Department of Mathematics, faculty of Science, Debark University, Ethiopia
b. Department of Mathematics Education, faculty of Science, UNISA, South Africa

Article Info
Received: January 7, 2024
Accepted: May 23, 2024
Published: July 30, 2024

10.46303/jcve.2024.13

How to cite
Belay, A. M., Machaba, F., & Makpakga, T. P. (2024). Exploring students' misconceptions in learning mathematical proof techniques at Debark University in Ethiopia. *Journal of Culture and Values in Education*, 7(2),70-100.
https://doi.org/10.46303/jcve.2024.13

ABSTRACT

This article focused on "Exploring students' misconceptions in learning mathematical proof techniques (MPT) at Debark University in Ethiopia." This research aimed to identify students' misconceptions in learning MPT and rank MPT based on the degree of students' misconceptions in learning MPT. Data were collected through assignments, follow-up, and structured interviews. Purposive sampling was used to select samples for follow-up interviews, whereas simple random sampling was used to select samples for structured interviews and assignments. The study collected data on the basis of mixed, case study, and pragmatic research approaches, designs, and paradigms from students. The results showed that the identified misconceptions of students in learning MPT were starting the proof with an inappropriate statement, using an ineffective MPT for their proof, providing incorrect symbolic representation, providing unacceptable reasons for each proof's step, reaching the conclusion without showing necessary steps clearly and neatly, showing non-sequential steps, incorrectly using technical aspects of mathematics, using the premise and conclusion parts interchangeably, incorrect perception regarding the technical concepts of proof, and misusing the pattern in the proof of a certain statement for the proof of another statement.

KEYWORDS

Exploring; mathematical proof; mathematical proof techniques; misconception; rank.

INTRODUCTION

This research focused on exploring students’ misconceptions in learning MPT at Debark University in Ethiopia. The research targeted third- and fourth-year students of the mathematics department at Debark University, which is located 830 km from the capital city of Ethiopia, Addis Ababa. In this study, students’ misconceptions in learning mathematical proof techniques were identified through assignments and interviews. In addition to this, the mathematical proof techniques were ranked on the basis of the degree of students’ Misconceptions in learning mathematical proof techniques.

Students at Ethiopia's lower-to-upper-level educational institutions have low academic performance in mathematics, according to Sileshi (2022), due to several factors, including learning misconceptions regarding the concepts of various mathematical disciplines. Specifically, according to Tamrat (2022), Debark University students perform poorly academically in mathematics, particularly in mathematical proof techniques, and they show no interest in choosing the mathematics department when it is offered. These were the driving forces behind the researchers’ decision to conduct a study titled "Exploring students’ misconceptions in learning mathematical proof techniques at Debark University in Ethiopia." Furthermore, as the title of the research indicated, this study contains two areas such as misconceptions (wrongly conceived or misconceived), and mathematical proof techniques (methods of mathematical proof).

LITERATURE REVIEW

The study took the two areas such as mathematical proof and learning misconceptions as literature. This section describes mathematical proof and learning misconceptions to identify students’ misconceptions in learning MPT at Debark University in Ethiopia and rank MPT based on the degree of students’ misconceptions in learning MPT.

Mathematical Proof

Mathematical proofing, according to Erickson and Lockwood (2021), is the process of proving a fact is true using a variety of methods and strategies in the study of mathematics. Menashe (2018), on the other hand, defines mathematical proof as an inferential argument that, given the stated assumptions (premise) of a mathematical statement, logically demonstrates the statement's conclusion. The skeleton of mathematical proof, according to Chamberlain and Vidakovic (2021), consists of three elements: a hypothesis, which is the statement's premise; a conclusion, which is the statement's result; and constructed ideas, which serve as a link between the mathematical statement's premise and conclusion.

According to Reiser (2020), mathematical proof techniques are methods that can be applied to establish the veracity of specific mathematical claims. Numerous methods exist for proving or disproving mathematical statements, as Miller et al. (2022) point out. Here are a few methods for proving or disproving a mathematical statement. Proof by contrapositive (PCP) can prove the given mathematical statement by starting with the contrapositive of the given statement (Levin, 2018). According to Emeira et al. (2020), proof by mathematical induction (PMI) proves the given mathematical statement using the principle of mathematical induction. Direct proof (DP) can take the premise of the given mathematical statement as the true statement to prove the truthfulness of the conclusion of the given statement (Tran, 2021). According to Rosen (2017), disproof by counterexample (DCE) disproves the given statement by thinking and taking counterexamples that falsify the given statement. Proof by construction (PCS) can prove the given mathematical statement by constructing counterexamples that satisfy the given statement (Jessica et al., 2020). According to Jessica et al. (2020), proof by exhaustion (PE) proves the given mathematical statement by taking a finite number of cases. Malinovsky (2022) states that a probabilistic proof (PP) proves the given mathematical statement using methods of probability theory. Proof by contradiction (PCD) can prove the given mathematical statement by starting the negation of the conclusion of the given statement (Hamdani et al., 2023). According to Mazur (2022), the combinatorial proof (CP) establishes the equivalence of different expressions by showing that they count the same object in different ways. Proof by using rules of inference (PRI) can use the rules of references for the proof of the mathematical statement, as Schauerhuber (2023) point out.

Learning Misconceptions

Misconception, according to Aschale et al. (2024), is the way of understanding or perceiving a certain concept incorrectly, denoting a certain concept incorrectly, constructing alternative incorrect definitions for a certain concept, and perceiving it as a correct and scientific definition. The phrases incorrect perception, understanding, notation, and definition express the word misconception (Jameson et al., 2023).

According to Fauziah and Muchyidin (2021), students' misconceptions in learning MPT mean that they can have incorrect ideas and understanding of proofs that are important to show the truthfulness of mathematical statements using different techniques in the learning of mathematics.

A disadvantage of misconceptions, as mentioned by Mathaba and Bayaga (2021), is that they lead to errors. If students have misconceptions about a certain concept, they lead to different types of errors because they have an incorrect understanding of the concept (Baselga & Olsen, 2021). Hence, a misconception is a cause of errors; however, errors are caused by misconceptions and other factors. Students' errors in the MPT have two forms: execution and conceptual errors, as Neidorf et al. (2020) point out. Gokkurt and Yenil (2023) state that execution error occurs when students do not use all procedures to execute the problem, whereas conceptual error occurs because of the failure to grasp the principles required for the solution of the problem or to recognize structural relationships in the problem.

Theoretical framework of this study

Jameson et al. (2023) state that the constructivist theory of learning is the theoretical framework of this study. It states that learners are not passive recipients of knowledge, but actively participate in the construction of their knowledge. They use their existing knowledge and relate it to new ideas, assuming meanings based on what they already know (Syukri et al., 2020). For example, in learning mathematics, learners use their past experiences as reference points, develop their thinking as they gain new experiences, and use these experiences to expand their knowledge base. This approach fosters a more active and effective learning process.

Objective of Study

This research aims to

- Identify students' misconceptions in learning MPT.
- Rank MPT based on the degree of students' misconceptions in learning MPT.

Research questions

This study was conducted to answer the following two research questions:

- What are the misconceptions of students regarding learning MPT?
- How can one rank MPT based on the degree of students' misconceptions in learning MPT?

METHODOLOGY

This study used a mixed research approach, a case study research design, and a pragmatism research paradigm. This study used assignments and interviews (follow-up and structured) to collect data from the third and fourth mathematics departments at Debark University in Ethiopia. In this department, there are 10 male and 4 female fourth-year mathematics department students, and 21 male and 5 female third-year mathematics department students.

To collect data from students through assignments, the study selected 9 male and 3 female students from fourth-year mathematics department students, and 15 male and 3 female students from third-year mathematics department students using simple random sampling.

To collect data from students through follow-up interviews, the study selected 4 male and 2 female students from fourth-year mathematics department students, and 3 male and 3 female students from third-year mathematics department students using purposive sampling.

To collect data from students through structured interviews, the study selected 3 male and 1 female students from fourth-year mathematics department students, and 6 male students from third-year mathematics department students using simple random sampling.

Quantitative data were analyzed and interpreted using various statistical methods, including frequency distribution tables, charts, and analysis of variance (ANOVA) using the Statistical Package for the Social Science (SPSS).

The following 10 standards in Table 1 were important for identifying students' misconceptions in learning mathematical concepts, as Safrtem (2021) and Ahmadpour et al. (2019) mentioned. Students' assignments and interviews were qualitatively and quantitatively analyzed and interpreted using the 10 standards to determine the rank of MPT based on the degree of students' misconceptions in learning MPT as well as identifying students' misconceptions in learning MPT in the case of Debark University.

Table 1.

Descriptions of the 10 Standards

Standards	Descriptions
Standard 1	Begin the proof with an inappropriate statement.
Standard 2	Select an ineffective MPT for proof of the statement.
Standard 3	Incorrect symbolic representation of the statement in the proof.
Standard 4	Provide unacceptable reasons for each step in the statement proof.
Standard 5	Conclude without showing the necessary steps clearly and neatly in the proof of the statement.
Standard 6	Show the non-sequential flow of steps in the proof of the statement.
Standard 7	Incorrect use of technical aspects of mathematics in the proof of the statement.
Standard 8	Interchangeable use of the premise and conclusion parts of a statement in its proof
Standard 9	Incorrect perception regarding the technical concepts of proof: axiom, theorem, corollary & lemma
Standard 10	Misuse of the pattern in the proof of a certain statement for the proof of another statement.

RESULTS AND DISCUSSION

Identified misconceptions from students' assignments and interviews

From the analysis of students' assignments considering standards stated by Safrtem (2021) and Ahmadpour et al. (2019), the students showed different misconceptions when they proved the mathematical statement in questions 1-18 of the assignment using MPT. To illustrate students' misconceptions in learning MPT, the study took twelve students' assignments. The students are described by their roll numbers such as SS12, SS29, SS3, SS14, SS7, SS21, SS2, SS13, SS18, SS9, SS15, and SS10 in the discussion stated below.

Identified misconceptions from SS12's assignments and follow-up interviews

Figure 1 shows SS12's proof for question 1 of the assignment. This student displayed various misconceptions (standards 1, 3, 5, and 9) in proving this question.

As stated in Figure 1, the student began with incorrect assumptions for the proof of question number 1. This means that the student expressed $\mathrm{a|b}$ and $\mathrm{b|(a+c)}$ as "$\mathrm{a = bk}$ for some integer k" and "$\mathrm{b = n(a+c)}$ for some integer n" respectively. These statements are incorrect because he has to express $\mathrm{a|b}$ and $\mathrm{b|(a+c)}$ as "$\mathrm{b = ak}$ for some integer k" and "$\mathrm{a + c = nb}$ for some integer n" respectively (Weingartner, 2022). Here, he displayed a misconception coded by standard 1, *beginning the proof with an inappropriate statement*.

Furthermore, the student showed the misconception as coded by standard 9, *incorrect perception regarding the technical concepts of proof, such as axiom, theorem, corollary, lemma, etc.* It is clear that SS12 does not know important technical concepts of the proof, such as the axiom, theorem, corollary, and lemma, that help to construct a proof. The student had incorrect concepts about factors. His incorrect perception regarding theorems of factors is a misconception coded by standard 9. From Figure 1, SS12 concluded $\mathrm{a|c}$ from $\mathrm{a = mc}$ and perception of $\mathrm{m} = \frac{\mathrm{kn}}{1-\mathrm{kn}}$ as an integer. From this, he had incorrect concepts regarding factors and integers because "if $\mathrm{a} = \mathrm{mc}$, then $\mathrm{c}|\mathrm{a}$" and "If $\mathrm{m} = \frac{\mathrm{kn}}{1-\mathrm{kn}}$ for integers k and n, and $\mathrm{kn} \neq 1$, then m is not always an integer" (Effinger & Mullen, 2021). His incorrect perception regarding theorems of factors and integers is a misconception coded by standard 9.

He concluded $\mathrm{c}|\mathrm{a}$ by taking an incorrect perception of $\mathrm{m} = \frac{\mathrm{kn}}{1-\mathrm{kn}}$ as integer always for integers k and n, and $\mathrm{kn} \neq 1$. Again, SS12 also displayed a misconception coded by standard 5, *providing a conclusion without showing the necessary steps clearly and neatly in the proof of the given statement*. It is evident in Figure 1 that SS12 reached an incorrect conclusion for the proof of the given statement, and the necessary steps were not clearly and neatly developed.

To confirm the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: How do you represent "a is a factor of b" symbolically?

SS12: $\mathrm{a}|\mathrm{b}$

Researcher: Express a|b in equation form.

SS12: $\mathrm{a} = \mathrm{kb}$ for some integer k.

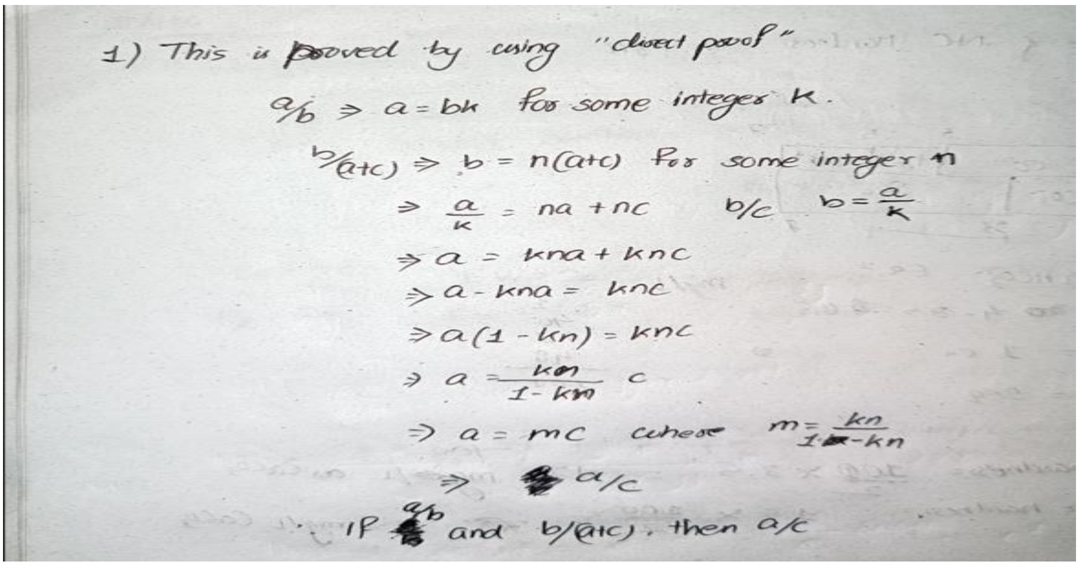
Researcher: Why did you conclude a|c from $\mathrm{a} = \frac{\mathrm{kn}}{1-\mathrm{kn}}\mathrm{c}$ in your proof of assignment question 1?

SS12: Because $\mathrm{m} = \frac{\mathrm{kn}}{1-\mathrm{kn}}$ can be an integer.

Moreover, as shown in the follow-up interviews, the student showed an incorrect symbolic representation of the statement in the interview (standard 3). In trying to use the correct symbolic representation of the statement in the proof to shorten and simplify the process of the proof, the student represents the given statement incorrectly during the time of constructing its proof, hence this misconception.

Figure 1.

SS12's Proof for Question 1 of the Assignment



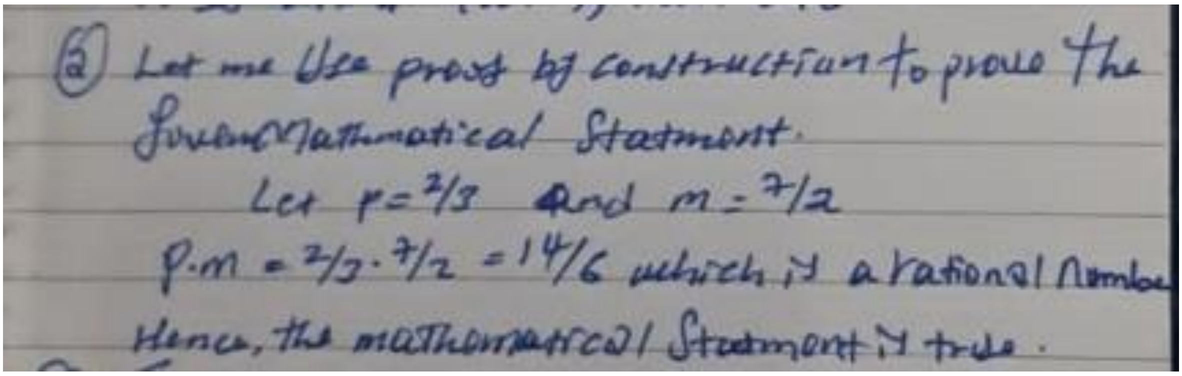
Due to the misconceptions held by the student on standards 1, 3, 5, and 9, he was unable to properly establish the proof of the mathematical statement in assignment question 1.

Identified misconceptions from SS29's assignments and follow-up interviews

The following figure shows SS29's proof for question 2 of the assignment. This student displayed various misconceptions (standards 2, 5 and 10) in proving this question.

Figure 2.

Proof of SS29 for Question 2 of Assignment


(2) Let me Use proof by construction to prove the [illegible] Mathematical Statment.
Let $p = 2/3$ and $m = 7/2$
$p.m = 2/3 . 7/2 = 14/6$ which is a rational Number
Hence, the mathematical Statment is true.

As stated in Figure 2, the student determined "Proof by construction" to prove the mathematical statement in question 2 of the assignment. Since the student did not understand the given mathematical statement and know that the word "every" in the mathematical statement affects the proof of it by selecting the appropriate MPT, he didn't determine the appropriate mathematical proof technique for the proof of question 2 because it is correctly proved using "Direct proof". Here, SS29 displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Again, SS29 also displayed a misconception coded by standard 5, *providing a conclusion without showing the necessary steps clearly and neatly in the proof of the given statement*. It is evident in Figure 2 that SS29 reached an incorrect conclusion for the proof of the given statement, and the necessary steps were not clearly and neatly developed.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: Which MPT is preferable to prove that a mathematical statement contains the word "every"?

SS29: Proof by construction.

Researcher: Which MPT is preferable to prove that a mathematical statement contains the word "some"?

SS29: Diproof by counterexamples.

Moreover, as shown in the follow-up interviews, the student considered an incorrect pattern to determine the MPT for the proof of mathematical statements that contain "every" and "some" (standard 10).

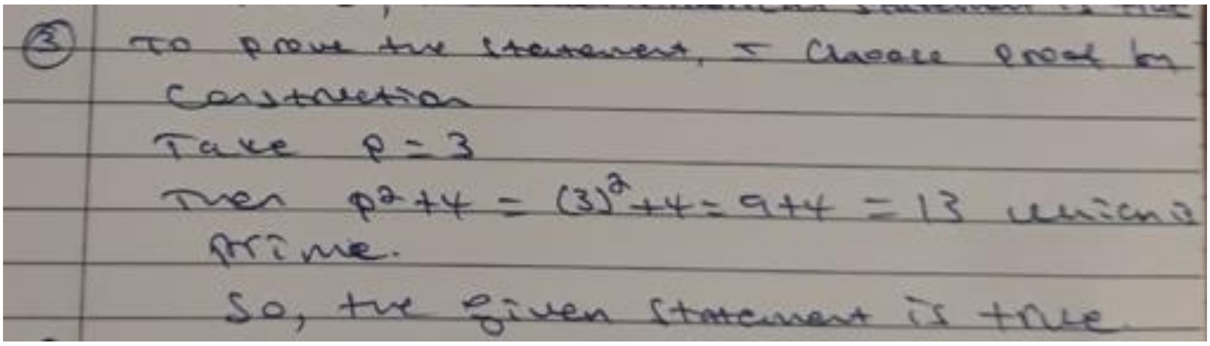
Due to the misconceptions held by the student on standards 2, 5, and 10, he was unable to properly establish the proof of the mathematical statement in assignment question 2.

Identified misconceptions from SS3's assignments and follow-up interviews

The following figure shows SS3's proof for question 3 of the assignment. This student displayed various misconceptions (standards 2, 5 and 10) in proving this question.

Figure 3.

Proof of SS3 for Question 3 of Assignment



As stated in Figure 3, the student determined "Proof by construction" to prove the mathematical statement in question 3 of the assignment. Since the student didn't understand the given mathematical statement and know that the word "every" in the mathematical statement affects the proof of it by selecting the appropriate MPT, she didn't determine the appropriate mathematical proof technique for the proof of question 3 because it is proved using "Disproof by counterexamples". Here, SS3 displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Again, SS3 also displayed a misconception coded by standard 5, *providing a conclusion without showing the necessary steps clearly and neatly in the proof of the given statement*. It is evident in Figure 3 that SS3 reached an incorrect conclusion for the proof of the given statement, and the necessary steps were not clearly and neatly developed.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: Is "Proof by construction" always preferable to prove that a mathematical statement contains the word "every"?

SS3: Yes

Researcher: Which MPT is preferable to prove that a mathematical statement contains the word "some"?

SS3: Diproof by counterexamples.

Moreover, as shown in the follow-up interviews, the student considered an incorrect pattern to determine the MPT for the proof of mathematical statements that contain "every" and "some" (standard 10).

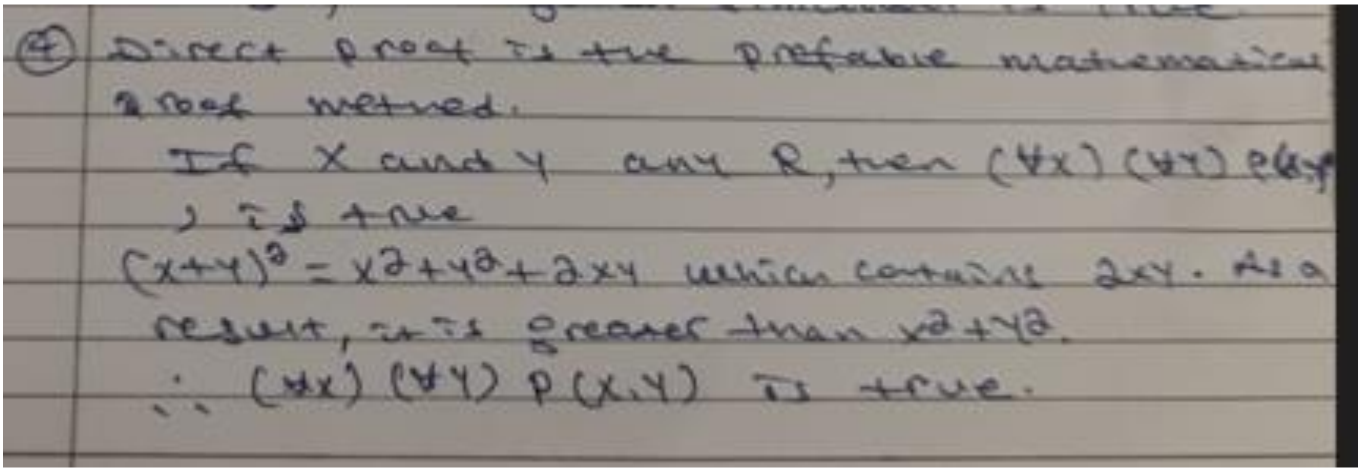
Due to the misconceptions held by the student on standards 2, 5, and 10, she was unable to properly establish the proof of the mathematical statement in assignment question 3.

Identified misconceptions from SS14's assignments and follow-up interviews

The following figure shows SS14's proof for question 4 of the assignment. This student displayed various misconceptions (standards 2, 4 and 5) in proving this question.

Figure 4.

Proof of SS14 for Question 4 of Assignment



(4) Direct proof is the prefable mathematical proof method.
If x and y any R, then $(\forall x)(\forall y)\, p(x,y)$ is true
$(x+y)^2 = x^2+y^2+2xy$ which contains $2xy$. As a result, it is greater than x^2+y^2.
$\therefore (\forall x)(\forall y)\, P(x,y)$ is true.

As stated in Figure 4, the student determined "Direct proof" to prove the mathematical statement in question 4 of the assignment. This is an incorrect determination because the mathematical statement can be proved correctly using "Disproof by counterexamples". Here, SS14 displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Furthermore, as shown in Figure 4, the student's reason why $x^2 + y^2 + 2xy$ is greater than or equal to $x^2 + y^2$ for any real number x and y is $x^2 + y^2 + 2xy$ contains 2xy but not $x^2 + y^2$. This is a false reason because $x^2 + y^2 + 2xy$ is less than or equal to $x^2 + y^2$ for negative real number x and positive real number y (Bordellès, 2022). Here, SS14 displayed a misconception coded by standard 4, *providing unacceptable reasons for each step in the statement proof*.

The correct proof of question 4 can be concluded by stating that the given mathematical statement is false. However, SS14 concluded the proof of question 4 by stating that the given mathematical statement is true. Here, SS14 displayed a misconception coded by standard 5, *concluding without showing the necessary steps clearly and neatly in the proof of the given statement*.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: Is always $x^2 + y^2 + 2xy$ is greater than or equal to $x^2 + y^2$?

SS14: Yes

Researcher: Why?

SS14: $x^2 + y^2 + 2xy$ contains more 2xy than $x^2 + y^2$.

Moreover, as shown in the follow-up interviews, the student provided an incorrect reason for "$x^2 + y^2 + 2xy$ is greater than or equal to $x^2 + y^2$". Here, SS14 displayed a misconception coded by standard 4, *providing unacceptable reasons for each step in the statement proof*.

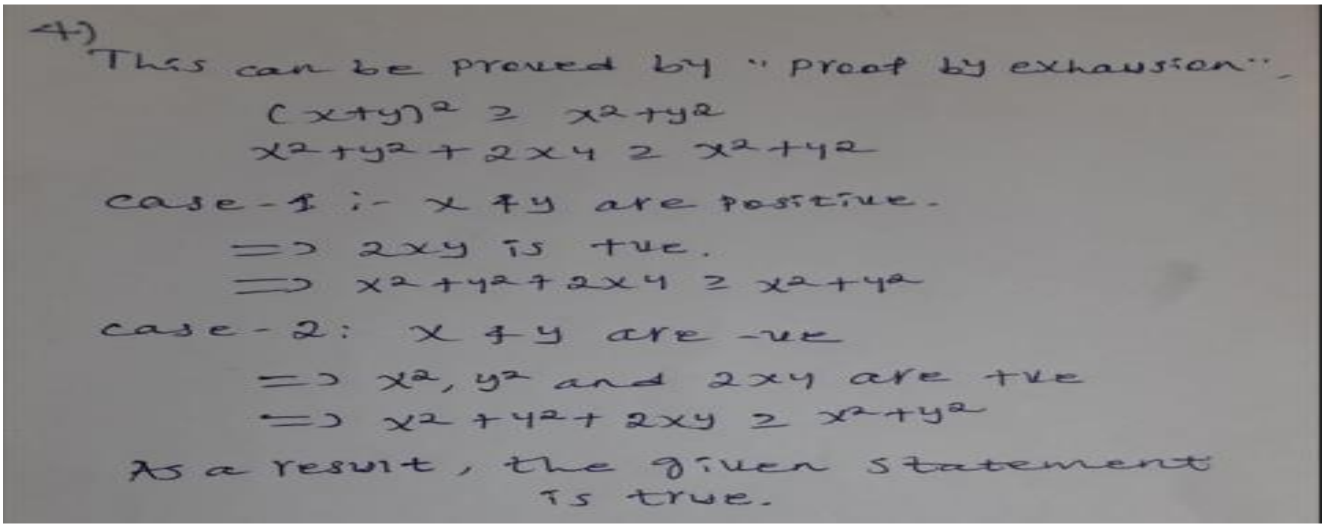
Due to the misconceptions held by the student on standards 2, 4, and 5, he was unable to properly establish the proof of the mathematical statement in assignment question 4.

Identified misconceptions from SS7's assignments and follow-up interviews

The following figure shows SS7's proof for question 4 of the assignment. This student displayed various misconceptions (standards 5 and 7) in proving this question.

Figure 5.

Proof of SS7 for Question 4 of Assignment



As stated in Figure 5, the student didn't show "Case 3: Let x and y have different signs" for the proof of question 4 of the assignment. This led the student to conclude the wrong conclusion because "If x and y have different signs, then $x^2 + y^2 + 2xy$ is less than or equal to $x^2 + y^2$ and the conclusion can be concluded as false" (Nagell, 2021). This is a misconception coded by standard 5, *concluding without showing the necessary steps clearly and neatly in the proof of the given statement*.

To check the misconception stated above, the follow-up interview of this student was conducted in the following ways.

Researcher: Why did you not consider $x < 0$ and $y > 0$ for the proof of question 4?

SS7: $x < 0$ is already included in Case 2 and $y > 0$ is already included in Case 1.

Moreover, as shown in the follow-up interview, the student misused the technical aspects of "$x < 0$ and $y > 0$" for cases "$x, y > 0$" and "$x,\ y < 0$". Here, SS7 displayed a misconception coded by standard 7, *incorrectly using the technical aspects of mathematics in the proof of the statement*.

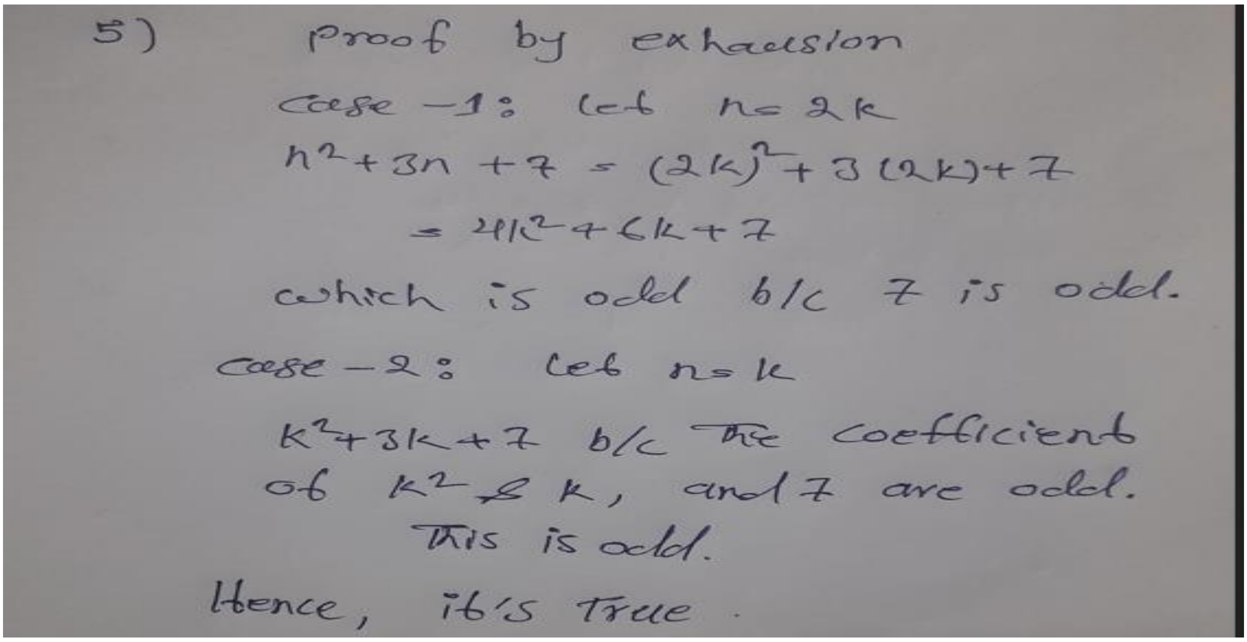
Due to the misconceptions held by the student on standards 5 and 7, he was unable to properly establish the proof of the mathematical statement in assignment question 4.

Identified misconceptions from SS21's assignments and follow-up interviews

The following figure shows SS21's proof for question 5 of the assignment. This student displayed various misconceptions (standards 1 and 4) in proving this question.

Figure 6.

Proof of SS21 for Question 5 of Assignment



As stated in Figure 6, the student gave the reason "7 is an odd" for $4x^2 + 6x + 7$ becomes an odd in case 1. This is not sufficient reason because he must state the reason as "4 and 6 are even, and 7 is odd" (Wahed, 2022). Here, SS21 displayed a misconception coded by standard 4, *providing unacceptable reasons for each step in the statement proof*.

Furthermore, as shown in Figure 6, the student gave the reason "coefficients of k^2 and k, and 7 in $k^2 + 3k + 7$ are odd" for $k^2 + 3k + 7$ becomes odd in case 2. This is the wrong reason because we can't determine any quadratic expression in the natural number n as odd by looking at its coefficients and constant terms (Balason, 2021). Here, SS21 displayed a misconception coded by standard 4, providing *unacceptable reasons for each step in the statement proof*.

Again, as described in Figure 6, the student considered $n = 2k$ for Case 1 and $n = k$ for Case 2. Considering $n = k$ for Case 2 is not correct because he has to consider $n = k + 1$ or $2k + 1$ for Case 2 (Gómez, 2021). Therefore, he started the proof in Case 2 with an incorrect statement. This is a misconception coded by standard 1, *beginning the proof with an inappropriate statement*.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: How do you determine "$ax^2 + bx + c$" is odd or not?

SS21: By looking at a, b, and c.

Researcher: Explain this

SS21: If a, b, and c are odd, then it is odd. If a, b, and c are even, then it is even. If at least one of a, b, and c is odd, then it is odd.

Moreover, as shown in the follow-up interviews, the student provided incorrect and unclear reasons for $ax^2 + bx + c$" is odd or not. Here, SS21 displayed a misconception coded by standard 4, *providing unacceptable reasons for each step in the statement proof*.

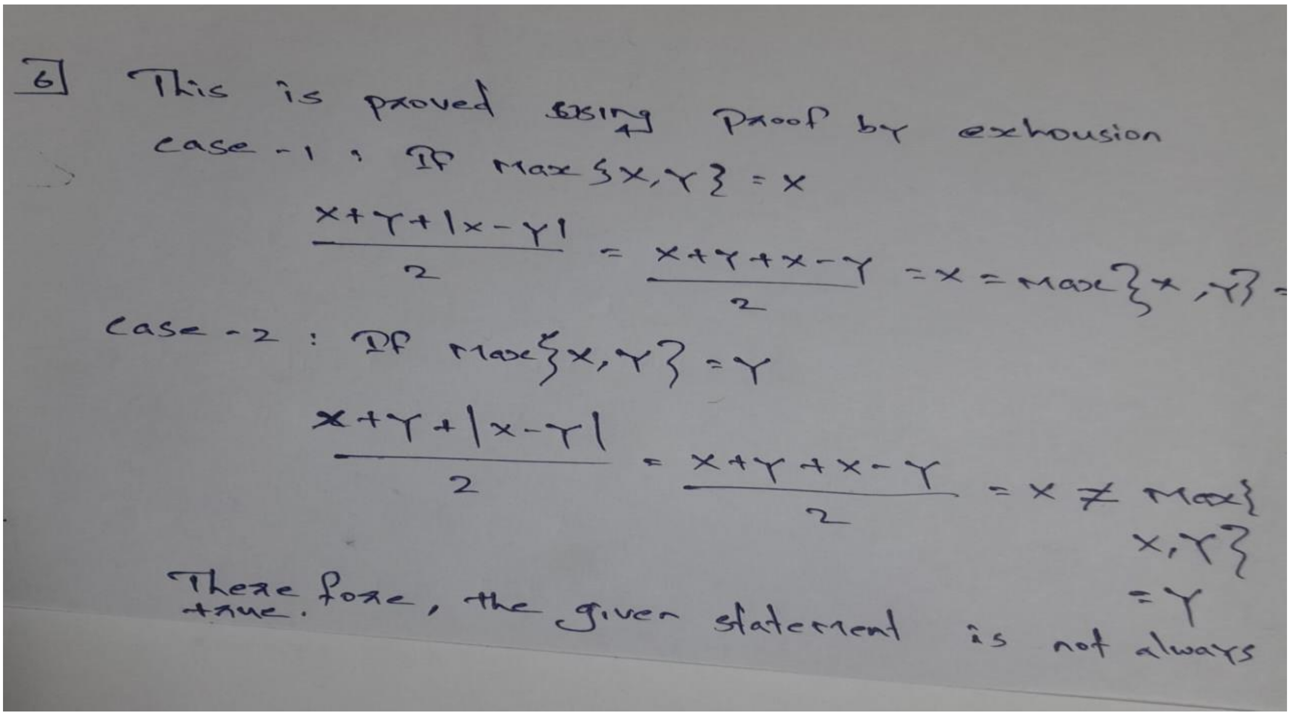
Due to the misconceptions held by the student on standards 1 and 4, he was unable to properly establish the proof of the mathematical statement in assignment question 5.

Identified misconceptions from SS2's assignments and follow-up interviews

Figure 7 shows SS2's proof for question 6 of the assignment. This student displayed various misconceptions (standards 1 and 7) in proving this question.

Figure 7.

Proof of SS2 for Question 6 of the Assignment



As stated in Figure 7, the student considered $\max\{x, y\} = x$ for case 1 and $\max\{x, y\} = y$ for case 2. These considerations are incorrect because she has to consider $x \geq y$ for case 1 and $y \geq x$ for case 2 (Roberts, 2014). These results showed that SS2 started the proofs with incorrect statements. Here, SS2 displayed a misconception coded by standard 1, *beginning the proof with an inappropriate statement*.

Furthermore, as described in Figure 7, the student substituted $x - y$ in place of $|x - y|$ which is found in the two cases. She used the concept of absolute value incorrectly. This is a misconception coded by standard 7, *incorrectly using technical aspects of mathematics in the proof of the statement*.

To check the misconception stated above, the follow-up interview of this student was conducted in the following ways.

Researcher: Is always $|x - y| = x - y$ true?
SS2: Yes.

Moreover, as observed in the follow-up interview, the student had an incorrect perception regarding absolute value. This is a misconception coded by standard 7, *incorrectly using technical aspects of mathematics in the proof of the statement*.

Due to the misconceptions held by the student on standards 1 and 7, she was unable to properly establish the proof of the mathematical statement in assignment question 6.

Identified misconceptions from SS13's assignments and follow-up interviews

Figure 8 shows SS13's proof for questions 7-9 of the assignment. This student displayed various misconceptions (standards 1, 2, 4, 5, 7 and 8) in proving these questions.

As stated in Figure 8, the student determined an inappropriate MPT to prove the mathematical statement in question 7 of the assignment because it is preferably proved using "Proof by contrapositive". This is a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Furthermore, as observed in Figure 8, the student used the premise and conclusion parts of the mathematical statement interchangeably in Question 7. This is a misconception coded by standard 8, *interchangeably using the premise and conclusion parts of a statement in its proof.*

As the student stated in his proof of question 7 in Figure 8, she provided a reason $a \geq 8 \text{ or } b \geq 8$ to conclude $a + b \geq 8 + 8 = 16$. Also, she provided a reason $15 \leq 16$ to conclude $a + b \geq 15 \geq 16$. These are incorrect reasons because "If $a \geq 8 \text{ or } b \geq 8$ for any integers a and b, then $a + b$ may or not be greater than 16" and "If $a + b \geq 16$, then we cannot insert 15 between $a + b$ and 16 i.e. $a + b \geq 15 \geq 16$" (Olmsted, 2018). Here, the student again displayed a misconception coded by standard 4, *providing unacceptable reasons for each step in the statement proof*.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

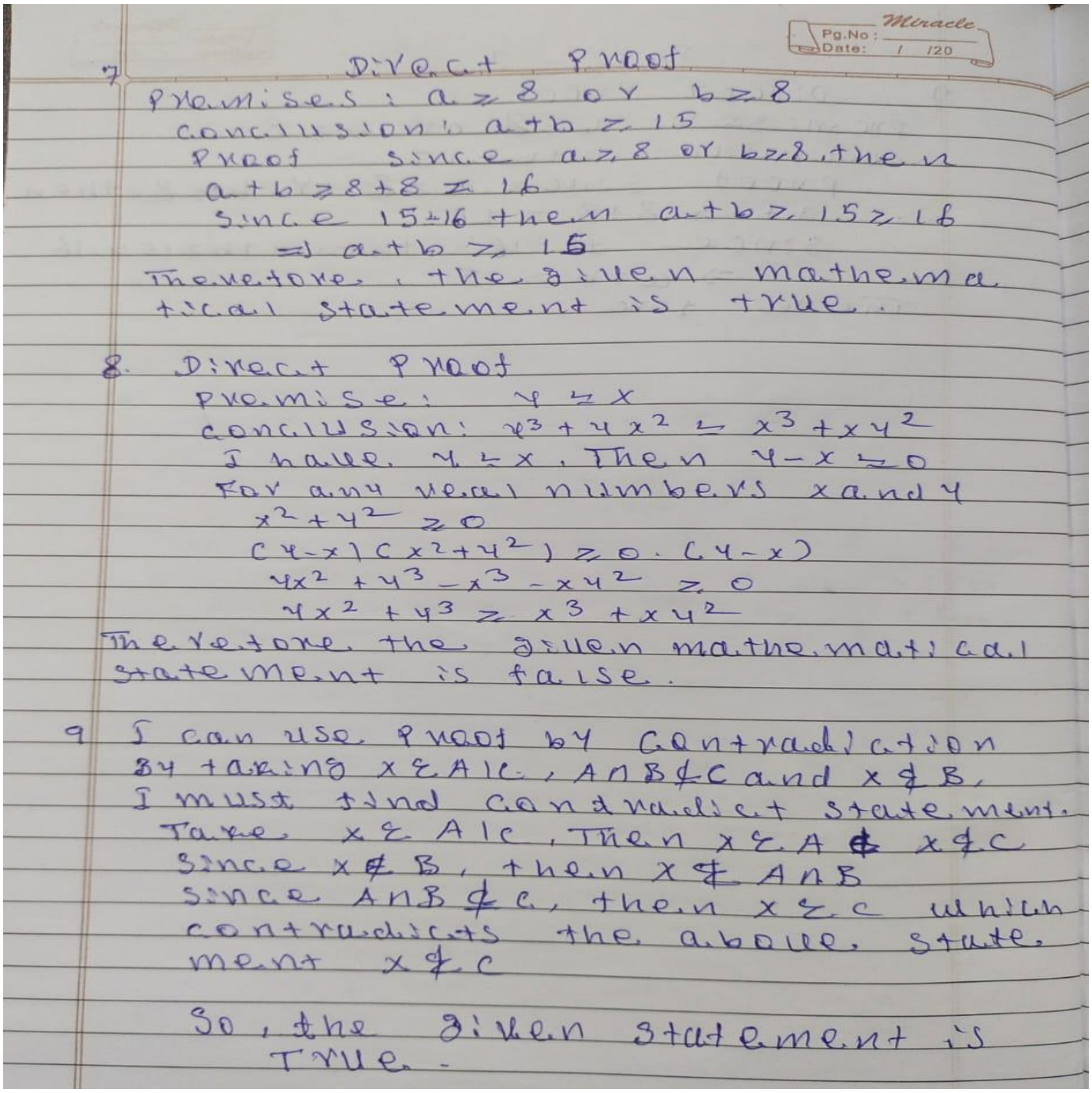
Researcher: Tell us the premise and conclusion of "If $a + b \geq 15$ for any integers a and $b, \text{then } a \geq 8 \text{ or } b \geq 8$".
SS13: The first statement, i.e. "$a + b \geq 15$ for any integers a and b" is the conclusion and the second statement, i.e., " $a \geq 8 \text{ or } b \geq 8$" is the premise.
Researcher: Is "If $a \geq 8 \text{ or } b \geq 8$ for any integers a and b, then $a + b \geq 16$" true?
SS13: Yes
Researcher: Is "If $a \geq 8 \text{ and } b \geq 8$ for any integers a and b, then $a + b \geq 16$" true?
SS13: Yes

Moreover, as shown in the follow-up interviews, the student did not know the technical aspects of mathematics, conjunction and disjunction. This is a misconception coded by standard 7, *incorrectly using technical aspects of mathematics in the proof of the statement*.

Due to the misconceptions held by the student on standards 2, 4, 7, and 8, she was unable to properly establish the proof of the mathematical statement in assignment question 7.

Figure 8.

Proof of SS13 for Questions 7–9 of the Assignment



As stated in Figure 8, the student determined an inappropriate MPT to prove the mathematical statement in question 8 of the assignment because it is preferably proved using "Proof by contrapositive". This is a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Furthermore, as observed in Figure 8, the student used the premise and conclusion parts of the mathematical statement interchangeably in question 8. This is a misconception coded by standard 8, *interchangeably using the premise and conclusion parts of a statement in its proof*.

As the student stated in his proof of question 8 in Figure 8, the student didn't consider the sign of $y - x$, i.e. $y - x \leq 0$ when she multiplied both sides of $x^2 + y^2 \geq 0$ by $y - x$. Here, the student displayed a misconception coded by standard 7, *incorrectly using technical aspects of mathematics in the proof of the statement*. As a result, SS13 reached a false conclusion, i.e. $yx^2 + y^3 \geq x^3 + xy^2$ (Sadler, 2019). She displayed a misconception coded by standard 5, *concluding without showing the necessary steps clearly and neatly in the proof of the given statement*.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways. The above-mentioned misconceptions were also observed in the follow-up interview.

Researcher: Tell us about the premise and conclusion of “$yx^2 + y^3 \leq x^3 + xy^2, \text{then } y < x$”.

SS13: First statement, i.e. “$yx^2 + y^3 \leq x^3 + xy^2$” is the conclusion and the second statement, i.e., “$y \leq x$” is the premise.

Researcher: Does the sign of c affect the inequality when you multiply both sides of $a \leq b$ by c for any real numbers a, b, and c?

SS13: No

Due to the misconceptions held by the student on standards 2, 5, 7, and 8, she was unable to properly establish the proof of the mathematical statement in assignment question 8.

As stated in Figure 8, the student started the proof of question 9 with an incorrect statement because if she determines the “Proof by contradiction” for the proof of question 9, she doesn’t use the negation of the premise part of question 9”. This means she has to use $A \cap B \subseteq C \text{ and } x \in B$ (Hamdani et al., 2023). Here, the student displayed a misconception coded by standard 1, *beginning the proof with an inappropriate statement*.

Furthermore, as shown in Figure 8, the student used $x \notin A \cap B \text{ and } A \cap B \nsubseteq C$ to conclude $x \in C$. If $x \notin A \cap B$, then $A \cap B \nsubseteq C$ doesn’t lead to say $x \in C$ because x may be outside of C (Jebril et al., 2021) and (Cenzer et al., 2020). Here, she provided a false reason for her proof (standard 4) and reached a false conclusion without showing the necessary steps clearly and neatly (standard 5).

To check the misconceptions stated above, the follow-up interview of this student was conducted in the following ways. The above-mentioned misconceptions were observed in the following follow-up interview.

Researcher: Can you use the negation of the premise for the proof of a mathematical statement using “Proof by contradiction”?

SS13: Yes. When I use “Proof by contradiction” for the proof of a mathematical statement, I must use the negation of its premise and conclusion.

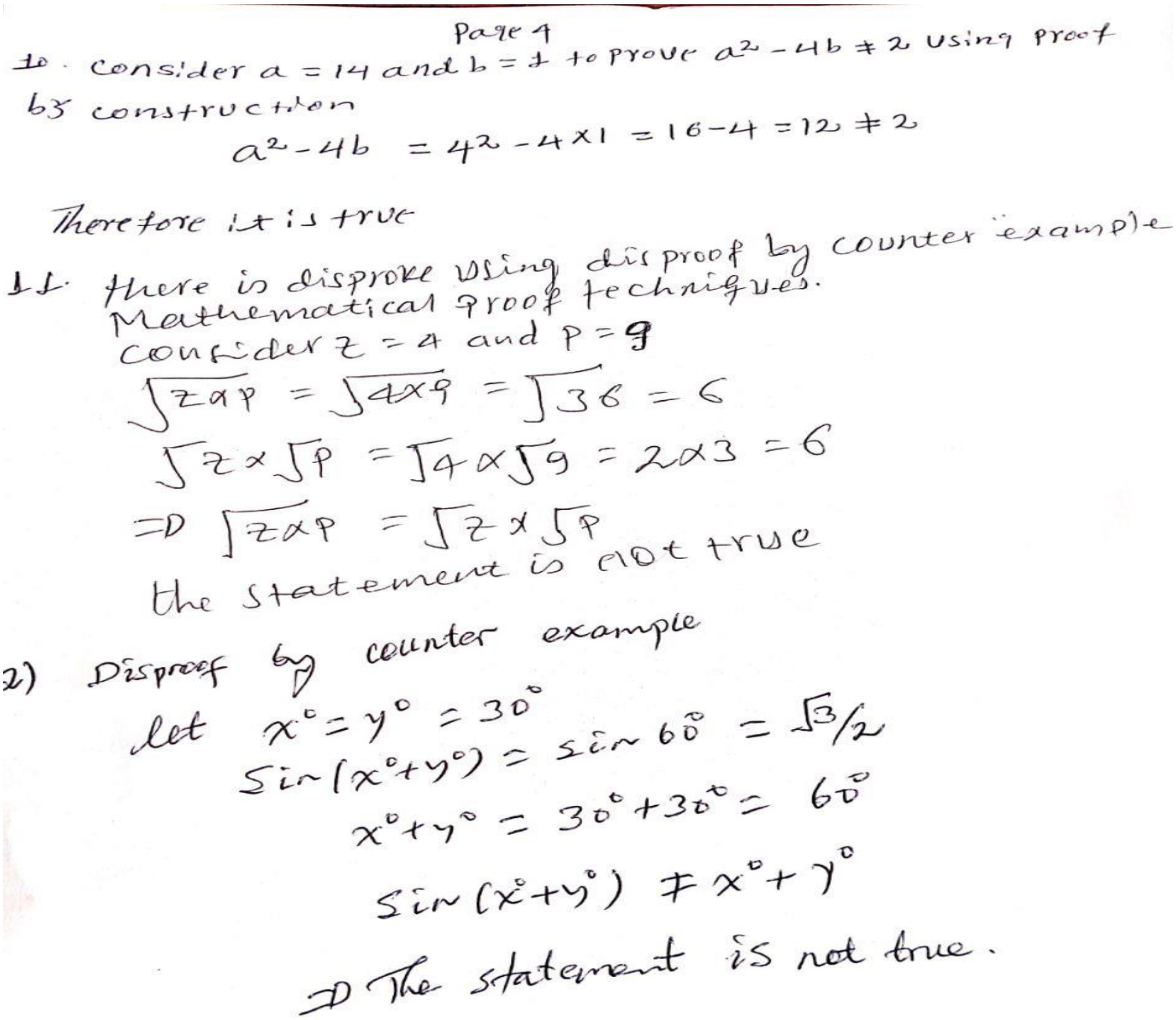
Due to the misconceptions held by the student on standards 1, 4, and 5, she was unable to properly establish the proof of the mathematical statement in assignment question 9.

Identified misconceptions from SS18’s assignments and follow-up interviews

Figure 9 shows SS18’s proof for questions 10–12 of the assignment. This student displayed various misconceptions (standards 2, 5 and 7) in proving these questions.

Figure 9.

Proof of SS18 for Questions 10-12 of the Assignment


Page 4

10. consider a = 14 and b = 1 to prove $a^2 - 4b \neq 2$ using proof by construction

$a^2 - 4b = 4^2 - 4 \times 1 = 16 - 4 = 12 \neq 2$

Therefore it is true

11. there is disprove using disproof by counter example Mathematical proof techniques.

consider z = 4 and p = 9

$\sqrt{z \alpha p} = \sqrt{4 \times 9} = \sqrt{36} = 6$

$\sqrt{z} \alpha \sqrt{p} = \sqrt{4} \alpha \sqrt{9} = 2 \alpha 3 = 6$

$\Rightarrow \sqrt{z \alpha p} = \sqrt{z} \alpha \sqrt{p}$

the statement is not true

12) Disproof by counter example

let $x^\circ = y^\circ = 30^\circ$

$\sin(x^\circ + y^\circ) = \sin 60^\circ = \sqrt{3}/2$

$x^\circ + y^\circ = 30^\circ + 30^\circ = 60^\circ$

$\sin(x^\circ + y^\circ) \neq x^\circ + y^\circ$

$\Rightarrow$ The statement is not true.

As stated in Figure 9, the student determined an inappropriate MPT for the proof of question 10 because it is preferably proved using "Proof by contradiction". Here, SS18 displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Furthermore, as shown, in Figure 9, the student didn't consider "any" when he proved the mathematical statement in question 10 of the assignment. In addition, he assigned 14 for a variable a, but he substituted 4 in place of a. This shows that he missed the technical aspect of mathematics. Here, the student showed a misconception coded by standard 7, incorrectly *using technical aspects of mathematics in the proof of the statement.*

Again, as observed in Figure 9, the student determined an inappropriate MPT for the proof of questions 11 and 12 without considering the difference between universal (for all, ∀) and existential (for some, ∃) quantifiers because it is preferably proved using "Proof by Construction" (Festa, 2020). Here, SS18 displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*. The student didn't know clearly the difference between universal (for all, ∀) and existential (for some, ∃) quantifiers (standard 7). As a result, he reached incorrect conclusions (standard 5).

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following way.

Researcher: Do you use “Proof by construction” for proofs of mathematical statements that contain “any”?

SS18: Yes.

Researcher: Do you use “Disproof by counterexamples” for proofs of mathematical statements that contain “there exists”?

SS18: Yes.

Moreover, as described in the interviews, the student had misconceptions about determining “Proof by construction” for the proof of a mathematical statement containing “any” and “Disproof by counterexamples” for the proof of a mathematical statement containing “there exists”. This is a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

Due to the misconceptions held by the student on standards 2, 5 and 7, he was unable to properly establish the proof of the mathematical statement in assignment questions 10–12.

Identified misconceptions from SS9’s assignments and follow-up interviews

Figure 10 shows SS9’s proof for question 13 of the assignment. This student displayed various misconceptions (standards 9 and 10) in proving this question.

As stated in Figure 10, the student wrote $C(n, k)$ and $C(n, k-1)$ as $\frac{n!}{k!}$ and $\frac{n!}{(k-1)!}$. This showed that the student had incorrect perceptions of the theorems of factor and combination (Hill, 2018). This is a misconception coded by standard 9, an incorrect *perception regarding the technical concepts of proof, such as axiom, theorem, corollary, and lemma.*

Furthermore, as shown in Figure 10, the student used the pattern in the theorem “If $a = c - b$ for any real numbers a, b, and c, then $a + b = c$” incorrectly for $C(n, k-1)$ to deduce $C(n+1, k)$ (Milanic et al., 2023). This is a misconception coded by standard 10, *misusing the pattern in the proof of a certain statement for the proof of another statement* because he used one pattern for another incorrectly.

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways. The above-mentioned misconceptions were observed in the following follow-up interviews.

Researcher: Express $C(n, k)$

SS9: $C(n, k) = \frac{n!}{k!}$

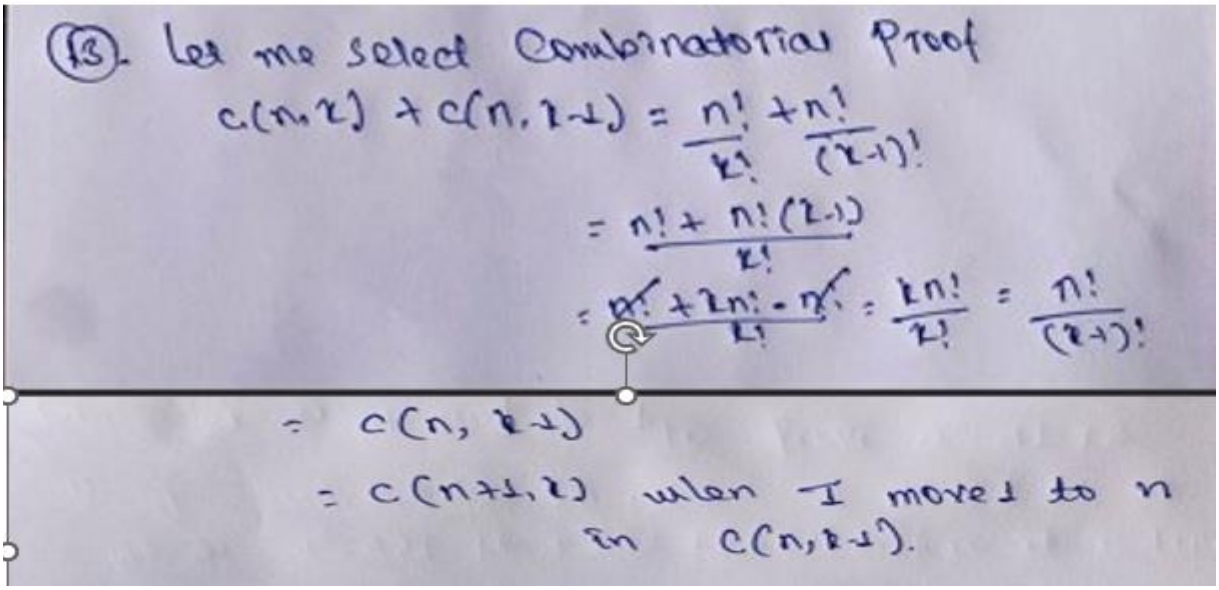
Researcher: Evaluate $C(6, 4)$

SS9: $C(6,4) = \frac{6!}{4!} = \frac{6x5x4x3x2x1}{4x3x2x1} = 6x5 = 30$

Due to the misconceptions held by the student on standards 9 and 10, he was unable to properly establish the proof of the mathematical statement in assignment question 13.

Figure 10.

Proof of SS9 for Question 13 of the Assignment



Identified misconceptions from SS15's assignments and follow-up interviews

Figure 11 shows SS15's proof for questions 14–16 of the assignment. This student displayed various misconceptions (standards 3, 5, 9 and 10) in proving these questions.

As stated in Figure 11, the student expressed the mathematical argument in question 14 in a wrong symbolic representation because he ignored the logical connective "Negation" in the mathematical argument for his symbolic representation. This is a misconception coded by standard 3, an *incorrect symbolic representation of the statement in the proof.*

Furthermore, as shown in Figure 11, the student selected "Proof by using rules of inferences" for the proof of question 14, but he used the pattern in "Tabular method" for his proof. This showed that he used the pattern of the "tabular method" for the proof of question 14 using "Proof by using the rule of inferences". Here, the student displayed a misconception coded by standard 10, *misusing the pattern in the proof of a certain statement for the proof of another statement.*

Because the mathematical statement in question 14 has three distinct statements, the constructed truth table must have 9 rows. However, the student constructed a truth table with six rows. Therefore, the student had an incorrect perception regarding the theorems of truth table construction (standard 9), and he concluded the proof without showing all steps clearly and neatly (standard 5).

To check the misconceptions stated above, the follow-up interviews of this student were conducted in the following ways.

Researcher: Express "If 3 is odd, then $3 + a$ is not always odd for any integer a.

SS15: $p \Rightarrow q$ where p: 3 is odd and q: 3+a is always odd for any integer a.

Researcher: Why do you not consider "not" in the conclusion of the mathematical statement when you represent it symbolically?
SS15: I didn't consider negation when I represented the mathematical statement symbolically.
Research: How many rows does "$(\mathrm{p} \Rightarrow \mathrm{q}) \wedge \mathrm{r}$" have?
SS15: $2\mathrm{x}3 = 6$ rows.

Moreover, as described in the follow-up interviews, SS15 misused the theorem "If a compound statement has n distinct statements, then its truth table has $2^{\mathrm{n}} + 1$ rows" (Jongsma, 2019). This is a misconception coded by standard 9, *perception regarding the technical concepts of proof, such as axiom, theorem, corollary, and lemma.*

Due to the misconceptions held by the student on standards 3, 5, 9, and 10, he was unable to properly establish the proof of the mathematical statement in assignment question 14.

As stated in Figure 11, the student didn't use the axioms of probability to prove question 16. this is a misconception coded by standard 9, an incorrect *perception regarding the technical concepts of proof, such as axiom, theorem, corollary, and lemma*.

Furthermore, the student started the proof at the mid-step of the proof for question 16. This showed that the student reached a conclusion without showing all the necessary steps of the proof clearly and neatly. This is a misconception coded by standard 5, *concluding without showing the necessary steps clearly and neatly in the proof of the given statement.*

To check the misconceptions stated above, the follow-up interview of this student was conducted in the following ways. The above-mentioned misconceptions were observed in the following follow-up interview.
Researcher: Is " $\mathrm{P(A \backslash B) = P(A) - P(A \cap B)}$ for any two events" a probability axiom?
SS15: Yes

Due to the misconceptions held by the student on standards 5 and 9, he was unable to properly establish the proof of the mathematical statement in assignment question 16.

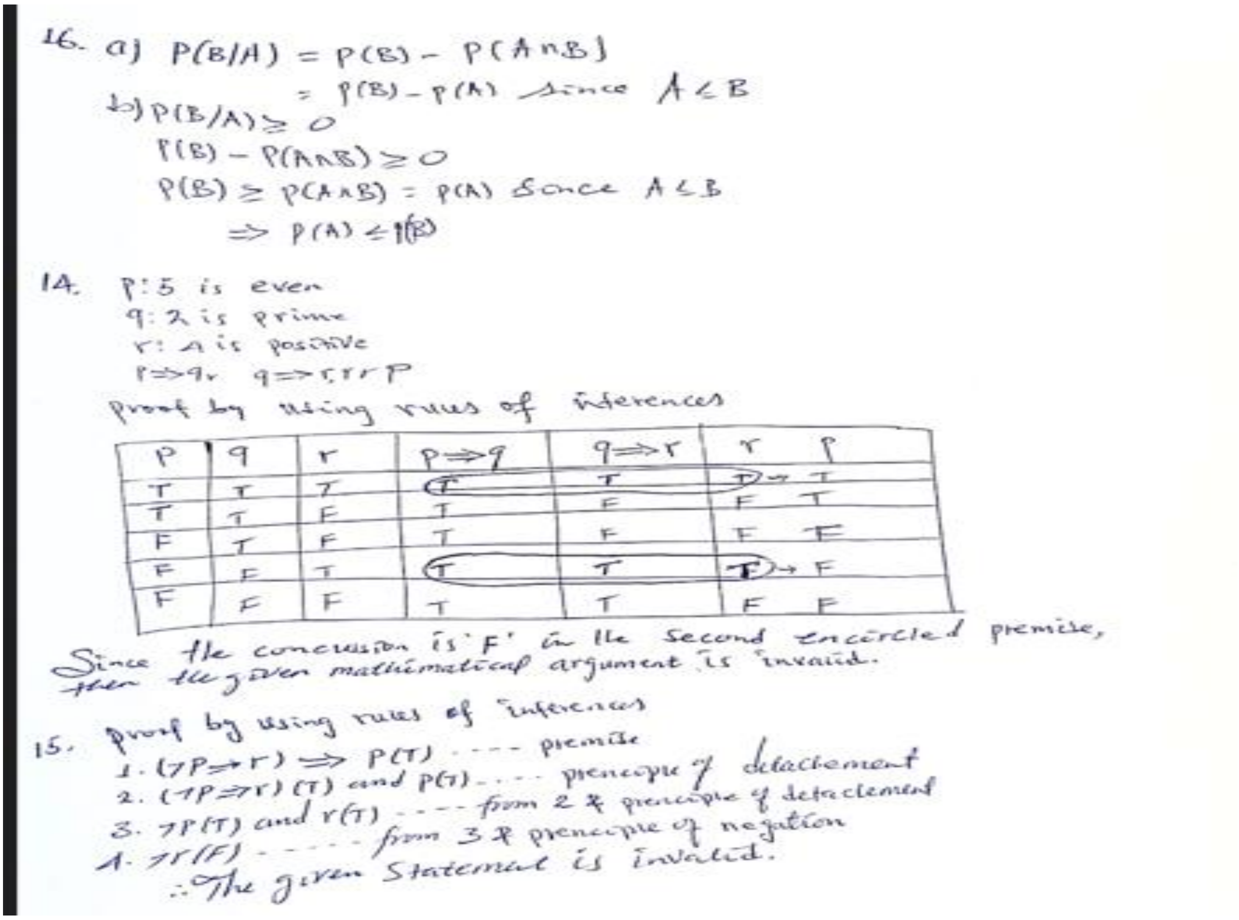
As stated in Figure 11, the student wrote incorrect statements in steps 2 and 3 of his proof for question 15 because he cannot use the principle of detachment to reach conclusions in steps 2 and 3 from $(\neg p \Rightarrow r) \Rightarrow p$ and $\neg p \Rightarrow r$ (Mileti, 2022). This showed that he used a pattern in the principle of detachment for dissimilar expressions (standard 10) and had incorrect perceptions of the principle of detachment (standard 9).

To check the misconceptions stated above, the follow-up interview of this student was conducted in the following ways. The above-mentioned misconceptions were observed in the following follow-up interview.
Researcher: Explain the principle of detachment.
SS15: If $\mathrm{p} \Rightarrow q$ is T, then both p and q are T.

Due to the misconceptions held by the student on standards 9 and 10, he was unable to properly establish the proof of the mathematical statement in assignment question 15.

Figure 11.

Proof of SS15 for Questions 14-16 of the Assignment



Identified misconceptions from SS10's assignments and follow-up interviews

Figure 12 shows SS10's proof for questions 17–18 of the assignment. This student displayed various misconceptions (standards 6 and 9) in proving these questions.

As stated in Figure 12, the student tried to prove questions 17 and 12 with a non-sequential flow of the proofs' steps. This is a misconception coded by standard 6, *showing the non-sequential flow of steps in the proof of the statement*. In addition, the student lacked clarity regarding the principle of mathematical induction (Papadopoulos & Paraskevi, 2021). This is a misconception coded by standard 9, an *incorrect perception regarding the technical concepts of proof, such as axiom, theorem, corollary, and lemma.*

To check the misconceptions stated above, the follow-up interview of this student was conducted in the following ways. The above-mentioned misconceptions were observed in the following follow-up interview.

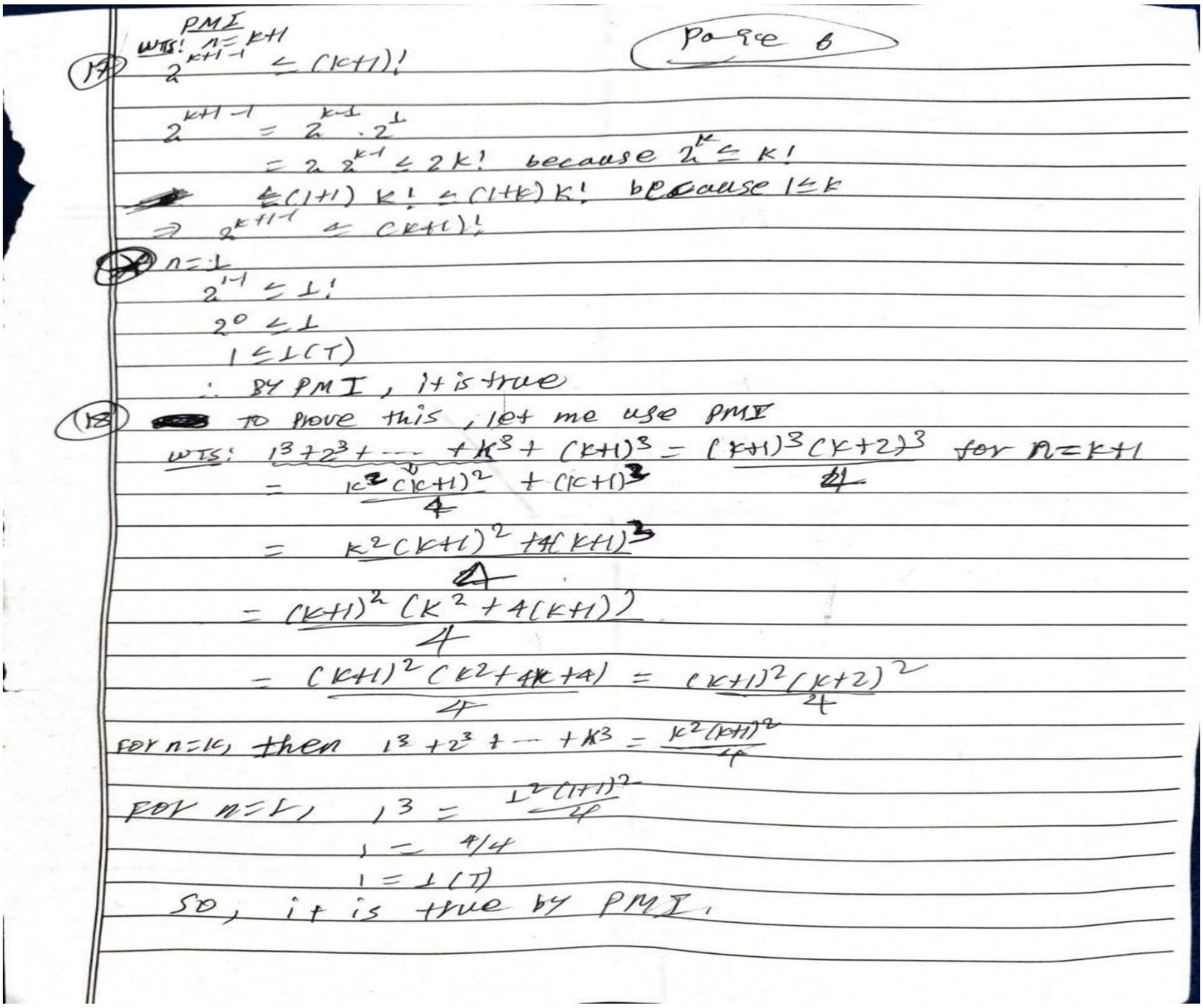
Researcher: State the three steps for proof of a mathematical statement using "Proof by mathematical induction".

SS10: In step 1, I must show that the mathematical statement is true for $n = k + 1$. In step 2, I must show that the mathematical statement is true for $n = k$. In step 3, I must show that the mathematical statement is true for $n = 1$.

Due to the misconceptions held by the student on standards 6 and 9, he was unable to properly establish the proof of the mathematical statement in assignment questions 17 and 18.

Figure 12.

Proof of SS10 for Questions 17-18 of the Assignment



Identified Misconceptions from Students' Structured Interviews

A structured interview for the students was conducted by ten students from the samples of the study to identify students' misconceptions in learning MPT. As stated in Appendix B, the interview has eight distinct questions.

In question 1 of the interview, students were mainly requested by the interviewer to state the difference between terminologies of mathematical proof such as axiom and theorem. Three students clearly stated the difference between the axiom and theorem. However, the remaining students didn't clearly state the difference between the axiom and theorem. Here, the students displayed a misconception coded by standard 9, *incorrect perception regarding the technical concepts of proof, such as axiom, theorem, corollary, and lemma.*

In question 2 of the interview, students were asked to state the premise and conclusion of the given mathematical statements. Two students among the interviewees made correct statements about the premises and conclusion parts of the given mathematical statement. However, the remaining students among the interviewees made incorrect statements about the premises and conclusion parts of the given mathematical statement by interchanging them. Here, the students displayed a misconception coded by standard 8, *interchangeably using the premise and conclusion parts of a statement in its proof*.

In question 3 of the interview, students were asked to represent a given mathematical statement in symbolic form. The interviewees showed minor and major errors while representing the given mathematical statement in symbolic form. Therefore, the students had a misconception coded by standard 3, *which is an incorrect symbolic representation of the statement in the proof.*

In question 4 of the interview, students were asked to list the MPT that are important for proving mathematical statements. The interviewees can list at least three MPT that are important for proving mathematical statements. Hence, students had no misconceptions in listing some MPT that are important for proving mathematical statements.

In question 5 of the interview, students were asked to determine the appropriate MPT for the proof of a given mathematical statement. Two students among the interviewees determined the appropriate MPT for the proof of a given mathematical statement. However, the remaining students among the interviewees determined an inappropriate MPT for the proof of the given mathematical statement. Therefore, these students displayed a misconception coded by standard 2, *selecting an ineffective MPT for proof of the statement*.

In question 6 of the interview, students requested to say whether their instructor invited them to participate actively in the teaching– learning processes of MPT or not. Because the courses at the University are very vast, the student said that instructors didn't invite them to participate actively in the teaching–learning processes of mathematics courses, including MPT. Hence, the teaching– learning methodology for learning MPT was not participatory.

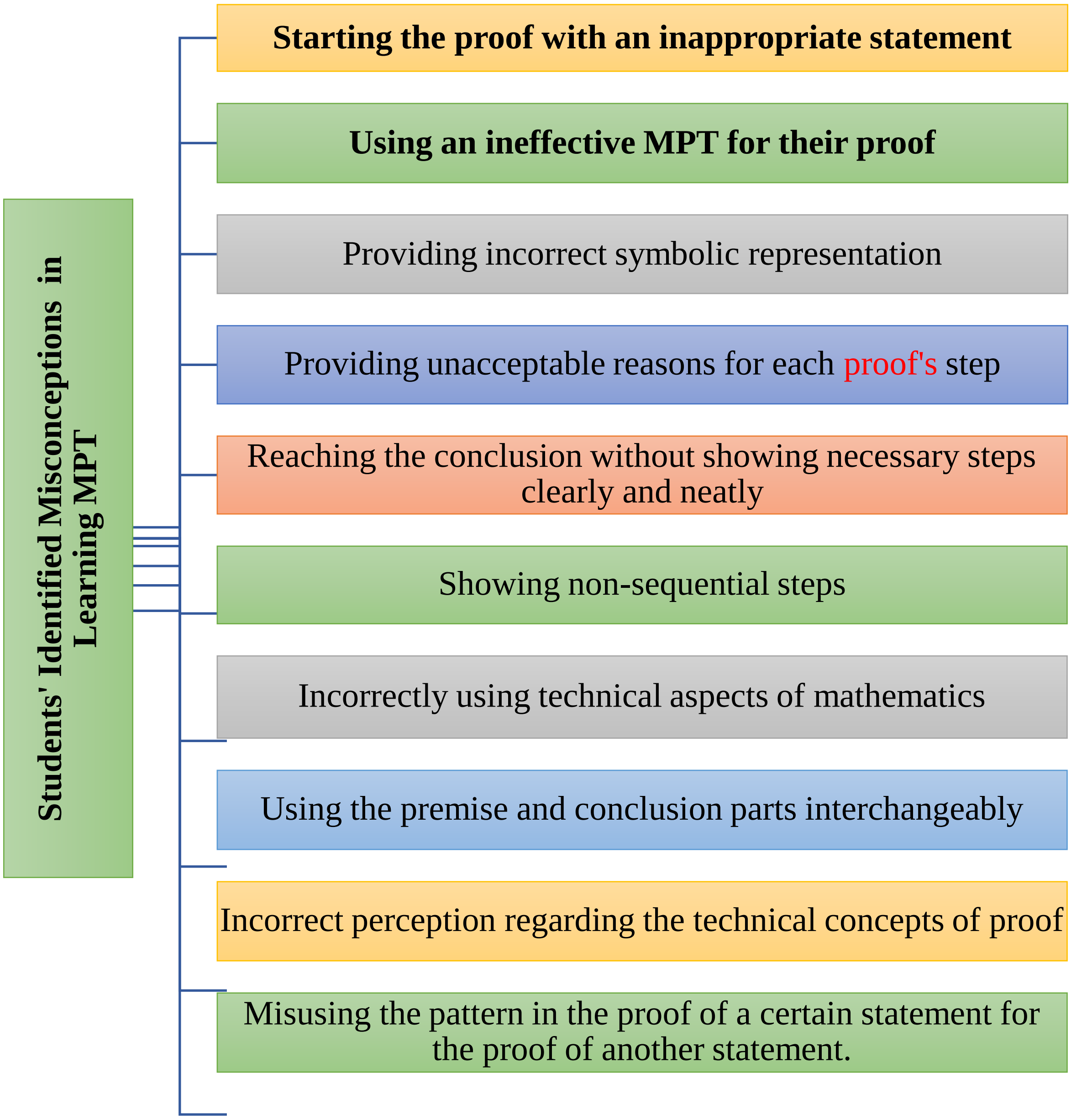
In question 7 of the interview, students were asked to state their beliefs and attitudes toward mathematical proof using MPT. The interviewees frankly stated that they feared mathematical proof and had negative attitudes toward mathematical proof using MPT. Hence, the students had negative attitudes and beliefs toward the mathematical proof of statements using MPT.

In the last question of the interview, the students were asked to list the advantages of mathematical proof using MPT. The interviewees frankly stated that mathematical proof has no advantages. It wastes both students' and instructors' time. Hence, the students believed that mathematical proof using MPT has no advantages.

Overall, from the above data, the identified students' misconception in learning MPT is described in the following figures shortly, which is the answer to RQ1.

Figure 13.

The Identified Students' Misconception in Learning MPT



Ranks of MPT based on students' misconceptions

To record the data in Table 3 using standards in Table 1, the study applied "if the question of the assignment shows standard 1, then assign 1 for a certain cell across a certain row of Table 2" or "if the question of the assignment doesn't show standard 1, then assign 0 for a certain cell across a certain row of Table 2". Note that a full description of the abbreviated words in Tables 2 and 3 is provided in the literature review section of this paper. Questions 1–2, 3–4, 5–6, 7–8, 9–10, 11–12, 13, 14-15, 16, and 17-18 of the assignment are proved by DP, DCE, PE, PCP, PCD, PCS, CP, PRI, PP, and PMI, respectively.

For instance, the misconceptions of students, who have roll number 1, in learning mathematical proof techniques were recorded in the students' portfolios just like as Table 2.

Table 2.

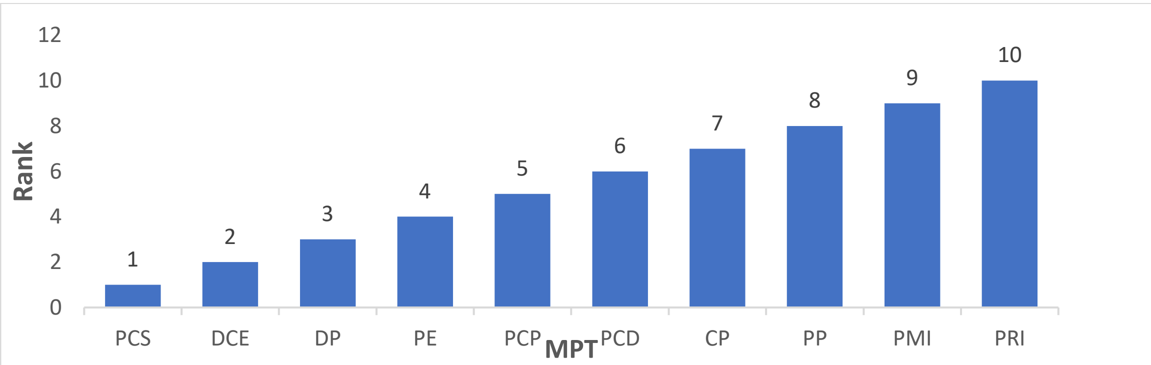
The Identified Misconceptions of Student (Roll No. One) in Learning MPT

Standards	DP	DCE	PE	PCP	PCD	PCS	CP	PRI	PP	PMI
1	0	1	0	1	0	1	0	1	0	1
2	1	0	0	0	0	0	0	1	1	0
3	1	0	1	0	1	1	1	1	1	1
4	1	1	0	1	0	0	1	0	0	0
5	1	0	1	0	0	1	1	1	1	1
6	0	0	1	0	0	0	0	0	0	0
7	0	0	0	1	0	1	1	0	1	1
8	1	0	1	0	1	0	0	1	0	1
9	0	1	0	0	1	1	0	1	1	1
10	1	0	1	0	1	1	1	1	1	1
Total	6	3	5	3	4	6	5	7	6	7

Aschale et al. (2024) state that third and fourth mathematics department students at Debark University in Ethiopia displayed different misconceptions in learning MPT with major differences among MPT. Students' assignments were administered by 30 students at Debark University to obtain detailed and primary data on students' misconceptions in learning MPT. The students' answers for each item were analyzed using standards, which are stated in Table 1 and Table 2, to rank MPT based on the degree of students' misconceptions in learning MPT. Numbers in Table 3 show the number of standards found in the items of the students' assignments. Under the age of Table 3 *(see Appendix C)* column, 1, 2, and 3 denote ages 18–23, 24–29, and 30–35.

Figure 14.

MPT Rank Based on the Degree of Students' Misconceptions in Learning MPT.



Dejen (2022) states that one-way ANOVA is performed to determine where there is a significant difference among more than two groups. Using one-way ANOVA, the p-value of the MPT used to prove mathematical statements in the students' assignment while considering students' identified misconceptions is 6.6E-12. Because the p-value of the MPT used to prove mathematical statements in the students' assignment while considering students' identified misconceptions is less than or equal to 0.05, there is a significant difference among the MPT used to prove mathematical statements in the students' assignment while considering students' identified misconceptions. Hence, the rank of MPT based on the degree of students' misconceptions in learning MPT can be determined using the mean of the number of students' identified misconceptions in each MPT.

Hence, the rank of MPT (from low severity to high severity) based on the degree of students' misconceptions in learning MPT is graphically stated as follows (answer of RQ 2).

CONCLUSIONS

The study found two results. The first result (answer of RQ 1) showed that the identified misconceptions of students in learning MPT are starting the proof with an inappropriate statement, using an ineffective MPT for their proof, providing incorrect symbolic representation, providing unacceptable reasons for each proof's step, reaching the conclusion without showing necessary steps clearly and neatly, showing non-sequential steps, incorrectly using technical aspects of mathematics, using the premise and conclusion parts interchangeably, incorrect perception regarding the technical concepts of proof, and misusing the pattern in the proof of a certain statement for the proof of another statement. The second result (answer of RQ 2) showed that the rank of the MPT in the context of Debark University based on the degree of students' misconceptions in learning MPT are PCS, DCE, DP, PE, PCP, PCD, CP, PP, PMI, and PRI.

Acknowledgements

We thank the Mathematics Department at Debark University in Ethiopia and UNISA in South Africa, which helped us to accomplish this study effectively and efficiently.

REFERENCES

Ahmadpour, F., Reid, D. & Reza, M. (2019). *Students' ways of understanding a proof.* Mathematical Thinking and Learning: 85–104.

Aschale, M., Machaba, F., & Makgakga, T. (2024). *Introducing a Supportive Framework to Address Students' Misconceptions and Difficulties in the Learning MPT: A Case of Debark University in Ethiopia.* Research in Social Sciences and Technology, 9(1), 63–84. https://doi.org/10.46303/ressat.2024.4.

Balason, N. (2021). *Quadratic Equations and Functions Workbook*. http://books.google.ie/books?id=tDyJzgEACAAJ&dq=quadratic+function&hl=&cd=2&source=gbs_api.

Baselga, S., & Olsen, M. (2021). *Approximations, Errors, and Misconceptions in the Use of Map Projections.* Mathematical Problems in Engineering, 2021, 1–12. https://doi.org/10.1155/2021/1094602.

Bordellès, O. (2022). On certain sums of number theory. *International Journal of Number Theory*, *18*(09), 2053–2074. https://doi.org/10.1142/s1793042122501056.

Cenzer, D., Larson, J., Porter, C., & Zapletal, J. (2020). *Set Theory and Foundations of Mathematics: An Introduction to Mathematical Logic - Volume I: Set Theory*. World

Scientific. http://books.google.ie/books?id=xNLkDwAAQBAJ&printsec=frontcover&dq=set+theory+and+logic+theory&hl=&cd=7&source=gbs_api.

Chamberlain, D., & Vidakovic, D. (2021). *Cognitive trajectory of proof by contradiction for transition-to-proof students*. The Journal of Mathematical Behavior, 62, 100849. https://doi.org/10.1016/j.jmathb.2021.100849.

Dejen, T. (2022). *Statistical package for social sciences (SPSS) for a researcher.* Ethiopia: Addis University Training and Consultancy Centre

Effinger, G., & Mullen, G. (2021). *Elementary Number Theory*. CRC Press. http://books.google.ie/books?id=b4k4EAAAQBAJ&printsec=frontcover&dq=elementary+number+theory&hl=&cd=10&source=gbs_api.

Emeira, G., Hapizah, & Scristia. (2020). *Mathematical proof analysis using mathematical induction of grade XI students.* Journal of Physics: Conference Series, 1480(1), 012044. https://doi.org/10.1088/1742-6596/1480/1/012044.

Erickson, S., & Lockwood, E. (2021). *Investigating undergraduate students' proof schemes and perspectives about combinatorial proof*. The Journal of Mathematical Behavior, 62, 100868. https://doi.org/10.1016/j.jmathb.2021.100868.

Fauziah, W., & Muchyidin, A. (2021). *misconception analysis in terms of student learning styles.* MaPan, 9(2), 197. https://doi.org/10.24252/mapan.2021v9n2a1.

Festa, R. (2020). *Mathematics, quantifiers, connectives, multiple models.* International Journal of Research in Science and Technology, 10(2), 31–64. https://doi.org/10.37648/ijrst.v10i02.005.

Gokkurt, B & Yenil, T. (2023). *Can Students' Misconceptions regarding Decimal Notation be Eliminated with the 5E Model Enriched with Digital Concept Cartoons?* International E-Journal of Educational Studies: 859–883.

Gómez, D. (2021). *Non-symbolic and symbolic number and the approximate number system.* Behavioral and Brain Sciences, 44. https://doi.org/10.1017/s0140525x21001175.

Hamdani, D., Purwanto, Sukoriyanto, & Anwar, L. (2023). *Causes of proof construction failure in proof by contradiction.* Journal on Mathematics Education, 14(3), 415–448. https://doi.org/10.22342/jme.v14i3.pp415-448.

Hill, T. (2018). *Essential Permutations & Combinations*. Create space Independent Publishing Platform. http://books.google.ie/books?id=hJe8tgEACAAJ&dq=combination+and+permutation++in+statistic&hl=&cd=1&source=gbs_api.

Jameson, G., Machaba, M., & Matabane, M. (2023). An Exploration of Grade 12 Learners' Misconceptions on Solving Calculus Problem: A Case of Limits. *Research in Social Sciences and Technology*, *8*(4), 94–124. https://doi.org/10.46303/ressat.2023.34.

Jebril, I. H., Dutta, H., & Cho, I. (2021). *Concise Introduction to Logic and Set Theory*. CRC Press. http://books.google.ie/books?id=9lE_EAAAQBAJ&printsec=frontcover&dq=set+theory+and+logic+theory&hl=&cd=4&source=gbs_api.

Jessica, C., Hapizah, & Scristia. (2020). *Analysis of student's proof construction on logarithms.* Journal of Physics: Conference Series, 1480(1), 012035. https://doi.org/10.1088/1742-6596/1480/1/012035.

Jongsma, C. (2019). *Introduction to Discrete Mathematics via Logic and Proof*. Springer Nature. http://books.google.ie/books?id=wEa9DwAAQBAJ&printsec=frontcover&dq=introduction+to+discrite+mathematics&hl=&cd=3&source=gbs_api.

Levin, O. (2018). *Discrete Mathematics*. Create space Independent Publishing Platform. http://books.google.ie/books?id=BWhCugEACAAJ&dq=a.%09Proof+by+contrapositive&hl=&cd=9&source=gbs_api.

Malinovsky, Y. (2022). *A Short Probabilistic Proof of a Binomial Identity*. The College Mathematics Journal, 53(5), 394–395. https://doi.org/10.1080/07468342.2022.2072093.

Mathaba, P., & Bayaga, A. (2021). *Analysis of Types, Sources of Errors and Misconceptions in South African Algebra Cognition*. Universal Journal of Educational Research, 9(5), 928–937. https://doi.org/10.13189/ujer.2021.090505.

Mazur, D. (2022). *Combinatorics*. American Mathematical Society. http://books.google.ie/books?id=HKatEAAAQBAJ&printsec=frontcover&dq=i.%09The+combinatorial+proof&hl=&cd=5&source=gbs_api.

Menashe, D. (2018). *
 Legal Proof, Mathematical Proof and Scientific Explanation*. SSRN Electronic Journal. https://doi.org/10.2139/ssrn.3132953.

Milanic, M., Servatius, B., & Servatius, H. (2023,). *Discrete Mathematics with Logic*. Elsevier. http://books.google.ie/books?id=fiS2EAAAQBAJ&printsec=frontcover&dq=discrete+mathematics&hl=&cd=5&source=gbs_api.

Mileti, J. (2022). *Modern Mathematical Logic*. Cambridge University Press. http://books.google.ie/books?id=5-WBEAAAQBAJ&printsec=frontcover&dq=mathematical+logic&hl=&cd=4&source=gbs_api.

Miller, D., Case, J., & Davies, B. (2022). *Students' beliefs on empirical arguments and mathematical proof in an introduction to proof class*. International Journal of Mathematical Education in Science and Technology, 1–22. https://doi.org/10.1080/0020739x.2022.2086082.

Nagell, T. (2021). *Introduction to Number Theory*. American Mathematical Soc. http://books.google.ie/books?id=Znc5EAAAQBAJ&printsec=frontcover&dq=number+theory&hl=&cd=4&source=gbs_api.

Neidorf, T., Arora, A., Erberber, E., Tsokodayi, Y., & Mai, T. (2020). *Student Misconceptions and Errors in Physics and Mathematics*. Springer.

http://books.google.ie/books?id=zu6rzQEACAAJ&dq=types+of+mathematical+errors+and+misconceptions&hl=&cd=2&source=gbs_api.

Olmsted, J. M. (2018). *The Real Number System*. Courier Dover Publications. http://books.google.ie/books?id=UitnDwAAQBAJ&printsec=frontcover&dq=real+number+system&hl=&cd=3&source=gbs_api.

Papadopoulos, I., & Paraskevi, K. (2021). *Reading Mathematical Texts as a Problem-Solving Activity: The Case of the Principle of Mathematical Induction*. Center for Educational Policy Studies Journal. https://doi.org/10.26529/cepsj.881.

Reiser, E. (2020). *Science Of Learning Mathematical Proofs, The: An Introductory Course*. World Scientific. http://books.google.ie/books?id=XEcREAAAQBAJ&printsec=frontcover&dq=techniques+of+Mathematical+proof&hl=&cd=6&source=gbs_api.

Roberts, C. (2014). *Introduction to Mathematical Proofs*. CRC Press. http://books.google.ie/books?id=dNEqBgAAQBAJ&pg=PA79&dq=proof+by+exhausion&hl=&cd=3&source=gbs_api.

Rosen, K. H. (2017). *Handbook of Discrete and Combinatorial Mathematics*. CRC Press. http://books.google.ie/books?id=Xj4PEAAAQBAJ&pg=PA58&dq=d.%09Disproof+by+counterexample&hl=&cd=1&source=gbs_api.

Sadler, R. (2019). *Pre-Calculus Workbook*. Carson-Dellosa Publishing. http://books.google.ie/books?id=ctuDDwAAQBAJ&pg=PA23&dq=trichotomy+property+of+real+numbers&hl=&cd=7&source=gbs_api.

Safrtem, G. (2021). *Analysis of students' Difficulties and Misconceptions appearing in students' tests and assignments.* Kenya: Pamlit pressing agency.

Schauerhuber, J. M. (2023). *Logical Rules of Inference and Replacement*. http://books.google.ie/books?id=8pIk0AEACAAJ&dq=rules+of+inference&hl=&cd=3&source=gbs_api.

Sileshi, F. (2022). *Problems of the Ethiopian Educational System between 1941-2000*. GRIN Verlag.

Syukri, A., Marzal, J., & Muhaimin, M. (2020). *Constructivism-Based Mathematics Learning Multimedia to Improve Students' Mathematical Communication Skills*. Indonesian Journal of Science and Mathematics Education, 3(2), 117–132. https://doi.org/10.24042/ijsme.v3i2.6201.

Tamrat, W. (2022). *Higher Education in Ethiopia*. BRILL.

Thierry, V. (2023). *Handbook of Mathematics*. BoD-Books on Demand. http://books.google.ie/books?id=1UrSEAAAQBAJ&pg=PA9&dq=f.%09Proof+by+exhaustion&hl=&cd=1&source=gbs_api.

Tran, Q. (2021). *A Direct Proof of Pitot's Theorem*. Mathematics Magazine, 95(1), 59–60. https://doi.org/10.1080/0025570x.2022.2001254.

Wahed, A. (2022). *number systems and their operations*. ShaShwat Publication. http://books.google.ie/books?id=rNOIEAAAQBAJ&printsec=frontcover&dq=properties+of+number+system&hl=&cd=9&source=gbs_api.

Weingartner, A. (2022). *The number of prime factors of integers with dense divisors*. Journal of Number Theory, 239, 57–77. https://doi.org/10.1016/j.jnt.2021.11.003.

APPENDICES

Appendix A: Format of students' assignment

Prove/disprove the following mathematical statements by selecting the relevant mathematical proof techniques, which are listed below, and showing the necessary steps clearly and neatly. Please keep in mind that the assignment's due date is ----------------------, and you exclude your name; rather, you must write your secured roll number.

Mathematical proof techniques: Combinatorial proof, direct proof, disproof by counter-examples, probabilistic proof, proof by construction, proof by contradiction, proof by contrapositive, proof by exhaustion, proof by mathematical induction, and proof by using rules of inference.

1. Let a, b, and c be integers. If $a|b$ and $b|(a + c)$, then $a|c$.
2. The product of two rational numbers is a rational number.
3. For any positive integer p, if p is prime, then $p^2 + 4$ is prime.
4. Let $P(x, y): (x + y)^2 \geq x^2 + y^2$, and x, y ∈ $\mathbb{R}$. Then $(\forall x)(\forall y)P(x, y)$ is true.
5. For any integer n, $n^2 + 3n + 7$ is odd.
6. For any real numbers x and y, $\mathrm{Max}\{x, y\} = \frac{x+y+|x-y|}{2}$
7. For any integer a and b, $a + b \geq 15$ implies $a \geq 8$ or $b \geq 8$.
8. Suppose $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$, the $y \leq x$.
9. Let $A \cap B \subseteq C$ and $x \in B$, then $x \notin A\backslash C$.
10. If a and b are any integers, then $a^2 - 4b \neq 2$.
11. There exist complex numbers z and p such that $\sqrt{z\,x\,p} \neq \sqrt{z}\,x\,\sqrt{p}$.
12. There exists x^0 and n^0 such that $\sin(x^0 + n^0) = x + n$.
13. Show that $C(n, k) + (n, k - 1) = (n + 1, k)$ where $0 < k \leq n$.
14. If 5 is even, then 2 is prime. 2 is prime if and only if 4 is positive. 4 is not positive. Therefore, 5 is not even.
15. $\neg p \wedge \neg q, (\neg q \Rightarrow r) \Rightarrow p \vdash \neg r$
16. If $A \subseteq B$, then

a. $P(B\backslash A) = P(B) - P(A)$

b. $P(A) \leq P(B)$.

17. For any natural number n, $2^{n-1} \leq n!$
18. For any natural number n, $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Appendix B: Format of a structured interview for students

1. Do you know the difference between an axiom and a theorem? If you say yes, state their differences.
2. What are the premise and conclusion parts of the mathematical statement "If x is between 3 and 4, then 3 is less than x and x is less than 4"?
3. Represent the mathematical statement in question 2 in symbolic form.
4. Do you know the MPT that are important for proving mathematical statements? If you say yes, list at least three MPT that are important to prove the mathematical statement.
5. Which MPT are relevant to the proof of "If n is even, 2n + 3 is odd"?
6. Do your teachers invite you to prove each mathematical statement in a course using MPT?
7. Do you like a course that invites you to prove each mathematical statement using MPT?

a. If you say yes, did you score a good result in the course?

b. If you say no, do you score low result in the course?

8. Do you know the benefits that can be derived from students learning a course by inviting them to prove each mathematical statement in the course using the mathematical proof technique? If you say yes, list the advantages.

Appendix C. Table 3.

Number of Standards Found in Items of the Assignment.

SS	Sex	Age	DP	DCE	PE	PCP	PCD	PCS	CP	PRI	PP	PMI
1	F	1	6	3	5	3	4	6	5	7	6	7
2	F	1	3	4	6	4	5	3	6	8	7	6
3	F	2	6	6	5	4	5	3	6	8	7	6
4	M	1	3	6	4	5	4	2	5	7	6	7
5	M	2	5	4	5	3	3	3	4	6	5	4
6	M	1	6	5	5	6	7	4	8	5	4	5
7	M	2	3	5	4	7	5	2	6	4	4	5
8	M	1	4	3	3	4	6	5	4	6	5	8
9	M	2	4	3	6	4	7	3	4	6	5	8
10	M	1	5	4	5	4	5	3	6	8	7	8
11	M	1	4	6	6	5	6	5	7	9	8	7
12	M	2	4	3	6	5	6	3	7	9	8	5
13	F	1	5	4	4	3	4	3	5	7	6	4
14	F	1	6	5	6	7	3	4	4	6	5	4
15	F	1	5	6	5	6	4	5	5	5	4	6
16	M	2	2	3	4	7	4	6	5	7	6	4
17	M	1	5	4	6	6	7	3	8	5	8	4
18	M	1	3	2	5	4	5	6	4	5	4	5
19	M	2	4	3	3	7	6	3	5	4	4	5
20	M	1	6	6	3	4	5	5	6	5	4	6
21	M	2	5	4	5	5	6	3	4	5	4	4
22	M	1	5	4	6	4	4	3	5	5	6	8

23	M	1	6	5	6	5	6	4	7	8	8	6
24	M	1	3	3	7	4	5	2	6	8	5	6
25	M	1	2	5	3	7	6	5	4	6	5	5
26	M	1	4	3	4	5	5	3	6	4	7	6
27	M	1	5	6	4	6	5	5	6	4	4	7
28	M	2	6	5	3	6	7	4	8	5	6	5
29	M	1	6	6	6	7	5	6	4	5	6	4
30	M	2	5	4	3	6	7	3	6	6	5	9
Mean			4.53	4.33	4.77	5.1	5.23	3.83	5.53	6.1	5.63	5.8
Rank			3rd	2nd	4th	5th	6th	1st	7th	10th	8th	9th